

Credit Valuation Adjustment (CVA)^{*†}

Shahram Alavian[‡]

Jie Ding[§]

Peter Whitehead[¶]

and

Leonardo Laudicina^{||}

March 22, 2010

Abstract

This paper provides an overview of counterparty valuation adjustments, within the context of collateralized and un-collateralized trading relationships. The counterparty valuation adjustment terms are derived by decomposing an un-defaultable portfolio into a set of binary states. These states are a set of market values of the portfolio (positive or negative), default states (default or no default) and recoveries (recover the recovery amount or not). In particular, the asset charge and liability benefit are formulated for both un-collateralized and collateralized portfolios while different models are provided for the collateral transfer calculations of the collateralized trading accounts.

*The opinions expressed are those of the authors and do not necessarily reflect the views of their employees, or other members of their staff.

†For questions and comments please contact Shahram Alavian at shahram_alavian@yahoo.com.

‡Lehman Brothers

§Nomura

¶Deutsche Bank

||Nomura

1 Introduction

According to the International Swap and Derivative Association (ISDA), the notional amount outstanding in over-the-counter (OTC) derivative trades, in the seven year period, between 2000 and 2007, has grown seven times [1]. Such an exponential growth in contractual commitments among financial entities demands a solid understanding and the proper evaluation of counterparty risk of the trading book.

Once two counterparties enter into a trade, besides market risk, they also take credit risk against each other. The risk of financial loss due to default of trading counterparties is referred to as counterparty credit default risk. In most cases, this risk is not considered in direct evaluation of the trades and, therefore, needs to be adjusted appropriately to reflect the risk should either of the counterparties default on their commitments. The adjustment to the value of a default free trading book is what is usually referred to as credit (or counterparty) valuation adjustment (CVA).

2 Governance of Trading Relationships

In most cases, in order to evaluate a default free trade, a trade specification is all that one needs. However, when considering the same trade with counterparty default considerations, one would also need to carefully examine the agreements under which the trading account is established. Since these agreements are a set of provisions that could be affecting all trades, they are usually not repeated in the individual trade confirms but they do, in a collective way, affect the value of the trades within the account.

In reality, there may be different agreements reached between any two counterparties. However, the ISDA Master Agreement introduced by ISDA is one of the most established agreements among most trading firms.

2.1 Trading Relationship Under ISDA Master Agreement

It is important to appreciate that though the counterparty ¹ with positive value of the portfolio is “in-the-money” with its trading counterparty, it is probably “out-of-the-money” with its hedging counterparty who is expecting the same incoming cash flow. Such a tight relationship, particularly among large dealers, creates a strong domino effect in defaults when one counterparty fails to deliver its commitment. For this reason dealers are encouraged

¹Specially in the case of a dealer

to find ways in order to reduce their counterparty risk [2]. The process of reduction in counterparty risk starts, prior to any trading activities, at the time when the trading agreement is to be signed. One of the most important documents to be signed by the counterparties regarding the governance of their trading relationship is the ISDA Master Agreement². The ISDA Master Agreement [3] has the objective of risk mitigation for both counterparties. Possibly the most important aspect of the ISDA Master Agreement is that the trades under this agreement form a single agreement. This is very important as it allows the parties to aggregate the amounts owing, by each of them, and replace them with a single net amount payable by one party to the other. Such agreement, therefore, enforces the counterparty risk to be evaluated on a portfolio level and not on a trade-by-trade basis. In general, where there are multiple ISDA Master Agreements, all trades under each agreement should form a single portfolio with no cross-netting between trades in different portfolios. To appreciate the effect of multiple master agreements, a good example to consider is the counterparty risk of two opposite but identical trades between two counterparties where each trade is booked under a different agreement. While this trade shows no market risk, due to asymmetry of exposure it is clearly subject to credit quality of both counterparties.

2.2 Trading Relationship Under Credit Support Annex

Though the ISDA Master Agreement reduces the risk by netting across all trades under its agreement, it still leaves a net residual exposure which, as the portfolio ages, can become substantial. Though for one portfolio one residual exposure just "maybe" substantial, for a dealer managing a large number of portfolios, the substantial exposures do exist. Since the dealer cannot easily sell the OTC trade to another client, it finds the exposure specific to the client; leaving the dealer still with more counterparty risk that it would like to have. Therefore, dealers are still inclined to have other mechanisms built in to their trading relationships under ISDA in order to minimize the residual counterparty risk even further. This is usually achieved by including an annex to their existing ISDA called Credit Support Annex (CSA). A CSA defines the terms under which collateral is posted or transferred between counterparties in order to mitigate counterparty credit risk³.

²For the purpose of consistency, all portfolios in this article are assumed to be covered, at the minimum, under an ISDA Master Agreement.

³Portfolios under CSA, due to the collateralized nature of the agreement, are called collateralized portfolios and those without CSA should be treated as

Terms in the CSA include thresholds, minimum transfer amounts, eligible securities and currencies, haircuts applicable to eligible securities and rules for the settlement of disputes arising over valuation of derivative positions. The threshold level is the credit that one counterparty extends to another, usually, conditional on its credit worthiness. The type and the haircut of the collateral are also predefined within the agreement. The minimum transfer amount is the buffer amount above the threshold to ensure some reasonable price movements occurs before the call for collateral is made. The main concept behind the CSA is the provisioning of the inflow of cash coming from the entity with out-of-the-money portfolio towards its trading counterparty. Depending on the direction of the flow of cash the agreement is of either bilateral or unilateral. In a bilateral agreement, which is the most common, both counterparties can call for collateral. In unilateral agreement, on the other hand, only one predefined counterparty has the right to call. As an example, consider the case of bilateral agreement with zero threshold and zero minimum transfer amount. This agreement essentially has the effect of constantly closing the book at the end of one period and reopening the same identical position at the beginning of the next; very similar to futures trading account. Therefore, in this case, the source of the exposure is the "change" of the portfolio value over and above the client's threshold since the last time collateral was exchanged. Another way of looking at the CSA bilateral agreement is through the concept of trade financing. In order to maintain a trade one needs money. This money can be financed by placing the position in a secure account as a collateral against the fund. This is a common practice in Repurchase Agreements (repo) except that in repo, OTC trades, since they are not transferable, are not accepted. In this case the counterparties themselves finance the in-the-money positions. In a bilateral agreement both counterparties finance each other while in the unilateral agreement the counterparty who has the right to call collateral enjoys the financing only. A simple and naive example will provide more insight. Imagine a client with a bilateral CSA agreement, covering zero threshold and zero minimum transfer amount for both counterparties, who is interested in selling a call to its dealer. In this case, the dealer, can keep the premium as a collateral tied up to the trade since the value of the premium is sufficient to cover the option. The dealer has just financed the client for its option. During the next margin period, if the value of the option, for whatever the reason, drops, the client's account now is in the money and it can now borrow the difference. However, if the option gains more value it would be the dealer who is in the money and the client would then lends cash to the dealer. Note that if the option

un-collateralized portfolios.

monotonically goes out of money until maturity, the client will end up collecting the entire premium of the option that it had sold without having to finance the trade.

On the calculation date which falls at the beginning of each margin period, the two counterparties evaluate their net positions and measure how far this value has changed from the last time any collateral has been transferred. The counterparty with the positive change, if it has the right to call for the collateral, will compare the change with the addition of its counterparty's threshold and the minimum transfer amount. If the value is lower, the exposure is within the credit line and no collateral should be transferred. If the value is higher, the counterparty with the positive change is entitled to receive collateral equal to the difference between the change and the threshold level; reducing its risk to the threshold level. This way, the counterparty maintains its exposure, maximum to the credit line it has granted, *regardless* of the value of the trade.

Therefore, if a counterparty entitled to the collateral does not receive the full payment during the grace period, the owed counterparty could mark the position as of the last evaluation date and would close both the trades and its corresponding hedges. It is important to note the implicit risk, to which the surviving counterparty is possibly exposed should the market moves deeply out-of-the-money with hedging positions while the actual trades are being closed and marked as of the calculation date. This is particularly of relevance when the number of OTC trades are large, since the larger the number of trades the more difficult it is to deal with complications arising from the close-out process. The close-out period can range from days (a small counterparty) to months (another dealer or broker).

3 Default Process Under ISDA Master Agreement

Roughly speaking, default is a credit event that allows one counterparty to record the last evaluation of the portfolio and initiate closing out the positions and pursue the owing counterparty for the value of the portfolio or the remaining of the collateral owed. Under ISDA, there are a number of situations defined as default. They are, failure to pay, breach of agreement, credit support default, default under special transactions, cross default, and, bankruptcy. One important trigger for the event is when the collateral payer does not transfer the collateral within a set period, called "grace period". Once the default occurs, the non-defaulting counterparty has the right (but

not the obligation) to serve the default notice, by hard copy, to allow the defaulting counterparty to remedy the situation, after which the termination date is automatically triggered. This grace period is usually a pre-defined in CSA, under which the defaulting counterparty can remedy the situation. This period depends on the type of ISDA, the products it covers and its governing jurisdiction. It usually falls some time within a month. Once the early termination is triggered, both counterparties resort to two methods of evaluating the net portfolio. For most liquid trades a third party dealer-quote and for most of the illiquid trades the replacement cost is used. In practice, it is not uncommon for the surviving counterparty to drastically inflate the positive-valued trades and deflate the negative-valued trades, and for the defaulting counterparty, under the administration, to dispute all values; making the recovery a long and indeterministic process that can drag on for years. Another commonly overlooked concept is that the surviving counterparty has the right but not the obligation to serve the default notice. If it decide not to proceed with the default notice, the trade remains live until maturity with no exchange of cash flow. Imagine a dealer where it has sold a series of options to a counterparty, under a single trade, for a premium. If the client defaults, the dealer may choose not to serve the notice and enjoy the cash in hand for as long as it finds it beneficial. In reality, the decision of not serving the default notice depends on many factors that may not even be based on maximizing the value of the trade. A surviving counterparty, with a few months before the end of its accounting year and a large exposure would probably decide to delay serving the default notice until the end of the accounting year, if the re-valuation of the trade would create a large write-down on its total asset. Regardless of when the notice is served, at the end of a long negotiation period and costly legal proceeding the surviving counterparty has to wait, as a unsecured bond holder, to be paid any recovery amount that the administration can provide. The larger the defaulting counterparty the longer the process of recovery.

The above paragraph is far from being the appropriate source for default proceedings. It is meant to provide a *feel* regarding what happens when a counterparty defaults under ISDA agreement. In the following paragraphs a set of commonly used formulae for valuing a trade, in the presence of the counterparty default, is derived. The advantage of deriving these formulae is to give insight to all the implicit assumptions and their relevance when applied to real practical applications.

4 Credit Valuation Adjustment

Consider the case where a LIBOR quality dealer enters into a back-to-back trade where it purchases a positive cash flow from a sub-LIBOR counterparty and sells an identical cash flow back to another LIBOR quality dealer for the same price. Without any counterparty risk consideration the dealer marks no profit/loss. However, in reality, this intermediary dealer has a loss due to the differential in counterparty risk that it maintained and did not pass to the other LIBOR quality dealer. To evaluate the adjustment, one common analogy to this differential value is to assume that the two counterparties implicitly sold each other an option to default on the trade. In the mentioned example, since the intermediary dealer has bought a positive cash flow, it has a positive receivable from the sub-LIBOR counterparty. Since it is at risk of counterparty's default, it has implicitly sold an option-to-default for which it never received the money. Therefore, the dealer should "charge" its trader the value of this option-to-default. On the other hand, the sub-LIBOR counterparty has implicitly bought the same option-to-default for which it never paid. Based on the same accounting principle, it should give its trader the "benefit" equal to the charge applied to the dealer's trader. Since the positive amount is represented as an asset on the dealer's trading book, the charge by the dealer to this underlying trade is commonly termed "asset charge". Based on the same token, the benefit given to the sub-LIBOR counterparty is called "liability benefit"⁴. Note that, in general, one's asset charge is its counterparty's liability benefit. In summary, asset charge is a receivable that was not charged and liability benefit is a payable that was not paid.

During a given period, either or both counterparties may default. Should that happen, the net exposure of the portfolio, at the time of default, and the portion of the amount to be lost determines the risk of the surviving counterparty. Therefore, in general, there are three contributing factors to counterparty risk: 1) credit states of the counterparties, 2) the credit risk free market value of the portfolio and 3) rates of recovery of both counterparties. To properly model the counterparty risk, all three should be viewed stochastic with their interdependence considered.

⁴The terms "asset charge" and "liability benefit" are chosen by preference only.

5 Decomposition of an Un-collateralized Portfolio

The motivation for this section is to formulate the CVA from the initial value of a portfolio which is not subject to counterparty default risk. This value is then expanded into a set of binary states with different cash flow directions (positive and negative), default states (default or not) and rate of recoveries (receive the rate of recovery or not) of both counterparties.

Under the pricing measure, the value of the portfolio, $V(t)$, as of time t , under the filtration \mathcal{F}_t with

$$V(t) = B(t, t')E[V(t')|\mathcal{F}_t] \quad (t' > t) \quad (1)$$

is considered. The bond price $B(t, t')$ defines the value of a risk free bond at time t maturing at t' . The filtration \mathcal{F}_t does include information on the counterparties' credit states. The trade may even be of credit derivative in nature depending on the credit states of verity of issuer names including any of the two counterparties as underlying market factors.

In order to understand the risk of a defaultable portfolio, one needs to know the conditional expectation of the portfolio in future time horizons. This is called "seasoning" or "aging" of the portfolio and it should consider the time dependency of the portfolio such as maturing underlying trades, in and outflow of cash flows as the portfolio marches along in to the future.

The common, and perhaps safe, interpretation of the ISDA Master Agreement regarding default is that at the time of either counterparties' default the market value of the portfolio is recorded as the balance owing and all trading activities stop. Therefore, one can model the seasoning of the portfolio only conditional on both counterparties' survival.

At this point, it is important to note that the definition of default does not necessarily mean the liquidation of the entity. According to ISDA, definition of "default" covers a number of states including restructuring. Therefore, the "first-time-to-default" is chosen since an entity may default more than once. In order to do this, one may define a random time τ as the time that the entity defaults for the first time. Note that for the purpose of this formulation, there is no need to condition τ to follow any process as long as it satisfies the first-time-to-default condition. For illustrative purposes, consider the counterparties *primed* and *unprimed*, each with first-time-to-default of τ and τ'^5 ,

⁵All primed and unprimed quantities would, hereafter, correspond to primed and unprimed counterparties. Later on, for the purpose of clarity, the CVA will be seen from the unprimed counterparty's point of view.

respectively.

Using the indicator function

$$\mathbb{I}_\theta \equiv \begin{cases} 1, & \text{if } \theta = \text{true} \\ 0, & \text{if } \theta = \text{false} \end{cases} \quad (2)$$

the values of the default free portfolio $V(t)$ can now be extended by both default states to time t as,

$$\begin{aligned} V(t) &= V(t|\tau' > t, \tau > t) + V(t|\tau' < t, \tau < t) + \\ &\quad V(t|\tau' < t, \tau > t) + V(t|\tau' > t, \tau < t) \\ &= V_s(t, \tau, \tau') + V_d(t, \tau, \tau') \end{aligned} \quad (3)$$

where the surviving portfolio $V_s(t, \tau, \tau')$ and the defaulted portfolio $V_d(t, \tau, \tau')$ are defined as

$$V_s(t, \tau, \tau') \equiv V(t) [\mathbb{I}_{\tau' > t} \cdot \mathbb{I}_{\tau > t}] \quad (4)$$

$$V_d(t, \tau, \tau') \equiv V(t) [\mathbb{I}_{\tau' < t} \cdot \mathbb{I}_{\tau < t} + \mathbb{I}_{\tau' < t} \cdot \mathbb{I}_{\tau > t} + \mathbb{I}_{\tau' > t} \cdot \mathbb{I}_{\tau < t}] \quad (5)$$

Conditional on both counterparties surviving until t , the surviving portfolio is one that is subject to risk while the defaulted portfolio, due to the indicator functions is null. The value of the surviving portfolio $V_s(t, \tau, \tau')$ can now be expanded to two binary states of incoming and outgoing cash flows as

$$\begin{aligned} V_s(t, \tau, \tau') &= \max[V_s(t, \tau, \tau'), 0] - \max[-V_s(t, \tau, \tau'), 0] \\ &\equiv V_s^+(t, \tau, \tau') - V_s^-(t, \tau, \tau'). \end{aligned} \quad (6)$$

Selecting an evaluation horizon, η , to be some δt further away from t .

$$\eta \equiv t + \delta t. \quad (7)$$

The forward default states, of both counterparties, can be chosen to fall before or after time η . One can expand unity into a complete set of default

states as

$$\mathbf{1}^+ = \mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau > \tau'} + \mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau < \tau'} + \mathbb{I}_{\tau' > \eta}$$

and

$$\mathbf{1}^- = \mathbb{I}_{\tau < \eta} \cdot \mathbb{I}_{\tau' > \tau} + \mathbb{I}_{\tau < \eta} \cdot \mathbb{I}_{\tau' < \tau} + \mathbb{I}_{\tau > \eta} \quad (8)$$

Note that for the continuous time axis the probability of mutual default is automatically null. Therefore, in the above equation, states for mutual defaults are not included. The right hand side (r.h.s) of either equations is the unity ($\mathbf{1}^\pm$) to be applied to $V_s^\pm(t, \tau, \tau')$ in (6), respectively. Note that the expansion here is different than what is provided in [4]. This is due to the fact that, here, the purpose is the fair valuation of the portfolio as opposed to calculating the exposure. From (6) and (8)

$$\begin{aligned} V_s(t, \tau, \tau') &= V_s^+(t, \tau, \tau') \mathbf{1}^+ - V_s^-(t, \tau, \tau') \mathbf{1}^- \\ &= V_s^+(t, \tau, \tau') [\mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau > \tau'} + \mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau < \tau'} + \mathbb{I}_{\tau' > \eta}] \\ &\quad - V_s^-(t, \tau, \tau') [\mathbb{I}_{\tau < \eta} \cdot \mathbb{I}_{\tau' > \tau} + \mathbb{I}_{\tau < \eta} \cdot \mathbb{I}_{\tau' < \tau} + \mathbb{I}_{\tau > \eta}] \end{aligned} \quad (9)$$

The next expansion can be done on the recovery states (recover the recovery amount or not) of both counterparties

$$\begin{aligned} V_s(t, \tau, \tau') &= V_s^+(t, \tau, \tau') [\mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau > \tau'} (1 - R') + \mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau > \tau'} R' + \\ &\quad \mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau < \tau'} + \mathbb{I}_{\tau' > \eta}] \\ &\quad - V_s^-(t, \tau, \tau') [\mathbb{I}_{\tau < \eta} \cdot \mathbb{I}_{\tau' > \tau} (1 - R) + \mathbb{I}_{\tau < \eta} \cdot \mathbb{I}_{\tau' > \tau} R + \\ &\quad \mathbb{I}_{\tau < \eta} \cdot \mathbb{I}_{\tau' < \tau} + \mathbb{I}_{\tau > \eta}] \end{aligned} \quad (10)$$

where R and R' are the rates of recovery for the primed and unprimed counterparties. As expected, the entire r.h.s of the above equation equals $V_s(t, \tau, \tau')$. Applying (4) to (10) and regrouping the terms allows one to finally obtain the value of a portfolio conditional on both counterparties survival until t broken in the following components

$$V_s(t, \tau, \tau') \tag{11}$$

=

$$+ V^+(t) \mathbb{I}_{\tau > t} \mathbb{I}_{\tau' > t} \cdot [\mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau > \tau'} (1 - R')] \tag{12}$$

$$- V^-(t) \mathbb{I}_{\tau > t} \mathbb{I}_{\tau' > t} \cdot [\mathbb{I}_{\tau < \eta} \cdot \mathbb{I}_{\tau' > \tau} (1 - R)] \tag{13}$$

$$+ V^+(t) \mathbb{I}_{\tau > t} \mathbb{I}_{\tau' > t} \cdot \left[\underbrace{\mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau > \tau'} R'}_{(1)} + \underbrace{\mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau < \tau'}}_{(2)} + \underbrace{\mathbb{I}_{\tau' > \eta}}_{(3)} \right] \tag{14}$$

$$- V^-(t) \mathbb{I}_{\tau > t} \mathbb{I}_{\tau' > t} \cdot \left[\underbrace{\mathbb{I}_{\tau < \eta} \cdot \mathbb{I}_{\tau' > \tau} R}_{(1)} + \underbrace{\mathbb{I}_{\tau < \eta} \cdot \mathbb{I}_{\tau' < \tau}}_{(2)} + \underbrace{\mathbb{I}_{\tau > \eta}}_{(3)} \right] \tag{15}$$

As defined during the course of this article, the positive cash flow represents inflow (received) and negative cash flow represents outflow (paid).

Assuming the positive value to be the incoming cash flow to the unprimed⁶ counterparty, $V^+(t) \mathbb{I}_{\tau > t} \mathbb{I}_{\tau' > t}$ and $V^-(t) \mathbb{I}_{\tau > t} \mathbb{I}_{\tau' > t}$ would represent the surviving asset and liability to the unprimed counterparty, respectively. The following interpretation of the above terms, from the unprimed counterparty's view, can be made⁷:

[11] Value of the portfolio without any counterparty risk considerations.

[12] Asset Charge: The portion, $1 - R'$, of the asset not received if the primed counterparty default first.

[13] Liability Benefit: The portion, $1 - R$, of the liability held if the unprimed counterparty default first.

[14.1] A portion, R' , of the asset received if the primed counterparty default first.

[14.2] Total amount of the asset received if the unprimed default first.

[14.3] Total amount of the asset received if the primed counterparty do not

⁶An easy, and perhaps crude, way to follow would also be to consider the unprimed counterparty to be "us" and the primed counterparty to be "them".

⁷It is understood that the definitions are made on the path measured by the filtration \mathcal{F}_t as defined in (1).

default during the evaluation horizon.

[15.1] A portion, R , of the liability paid if the unprimed counterparty default first.

[15.2] Total amount of the liability paid if the primed counterparty default first.

[15.3] Total amount of the liability paid if the unprimed counterparty do not default during the evaluation horizon.

One interpretation of the equations (12) through (15) can be the following: the contributions to the value of the surviving portfolio, $V_s(t, \tau, \tau')$ come from two sources of loss [(12) and (13)] and recovery⁸ [(14) and (15)], of both counterparties. Defining $V_s^*(t, \tau, \tau')$ to be the net total of recovery values of all cash flows, between t and η , conditional on both counterparties surviving until t ,

$$\begin{aligned} V_s^*(t, \tau, \tau') \equiv & \\ & +V^+(t) \mathbb{I}_{\tau > t} \mathbb{I}_{\tau' > t} \cdot [\mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau > \tau'} R' + \mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau < \tau'} + \mathbb{I}_{\tau' > \eta}] \\ & -V^-(t) \mathbb{I}_{\tau > t} \mathbb{I}_{\tau' > t} \cdot [\mathbb{I}_{\tau < \eta} \cdot \mathbb{I}_{\tau' > \tau} R + \mathbb{I}_{\tau < \eta} \cdot \mathbb{I}_{\tau' < \tau} + \mathbb{I}_{\tau > \eta}] \end{aligned} \quad (16)$$

One can now define CVA as the adjustment to the surviving portfolio in order to obtain the value of the portfolio with counterparty risk $V_s^*(t, \tau, \tau')$.

$$\mathbf{CVA}(t, \tau, \tau') \equiv V_s^*(t, \tau, \tau') - V_s(t, \tau, \tau') \quad (17)$$

$$-V^+(t) \mathbb{I}_{\tau > t} \mathbb{I}_{\tau' > t} \cdot [\mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau > \tau'} (1 - R')] \quad (18)$$

$$+V^-(t) \mathbb{I}_{\tau > t} \mathbb{I}_{\tau' > t} \cdot [\mathbb{I}_{\tau < \eta} \cdot \mathbb{I}_{\tau' > \tau} (1 - R)] \quad (19)$$

Equation (18) and (19) are the components of CVA conditional on the filtration \mathcal{F}_t . The term $V_d(t, \tau, \tau')$ was omitted in the above equation for two reasons. One reason is that CVA is based on the condition that both counterparties survive until t , the time of evaluation. The second reason why it was omitted is that the recovery was assumed realized at the time of default. Should the recovery occur after the default event (which is usually the case), then the defaulted portfolio, $V_d(t, \tau, \tau')$, should be included in the model. In

⁸Note that in the equations (14.2), (14.3), (15.2) and (15.3) the rate of recovery can be interpreted as 100%.

cases where the default has occurred, the recovery may be in the future and needs to be accounted for.

5.1 Derivation of Asset Charge

The more practical representation of the asset charge for the un-collateralized portfolio is shown here. Starting from (12), define

$$AC_u(t; \tau, \tau') \equiv V^+(t) \mathbb{I}_{\tau > t} \mathbb{I}_{\tau' > t} \cdot [\mathbb{I}_{\tau' < \eta} \cdot \mathbb{I}_{\tau > \tau'} (1 - R')] \quad (20)$$

Equation (20) represents the stochastic value of asset charge for one single continuous period between t and η as of time t . If the entire length of the trade is divided into N number of evaluation horizons, each δt infinitesimally long, with

$$i = 0, 1, 2, \dots, N - 1 \quad (21)$$

$$AC_u(T_i | \mathcal{F}_{T_i}) \equiv E [V^+(T_i) \mathbb{I}_{\tau > T_i} \mathbb{I}_{\tau' > T_i} \cdot \mathbb{I}_{\tau' < T_{i+1}} \mathbb{I}_{\tau > \tau'} (1 - R') | \mathcal{F}_{T_i}] \quad (22)$$

would represent the asset charge contribution of each period measured at the beginning of each period conditional on both counterparties surviving until T_i . Incorporating the conditional survival,

$$AC_u(T_i | \mathcal{F}_{T_i}) = E [V^+(T_i) \mathbb{I}_{\tau' \in (T_i, T_i + \delta t)} \cdot \mathbb{I}_{\tau > \tau'} (1 - R') | \mathcal{F}_{T_i}] \quad (23)$$

The unconditional expectation of the asset charge is

$$AC_u(T_i) = B(0, T_i) E [V^+(T_i) \mathbb{I}_{\tau' \in (T_i, T_i + \delta t)} \cdot \mathbb{I}_{\tau > \tau'} (1 - R')] \quad (24)$$

where

$$E[\cdot] \equiv E[\cdot | \mathcal{F}_0] \quad (25)$$

and (1) were used. Finally the total contribution, valued at time 0, is

$$\begin{aligned}
AC_u &= \sum_{i=0}^{N-1} AC_u(T_i) \\
&= \sum_{i=0}^{N-1} B(0, T_i) E \left[V^+(T_i) \mathbb{I}_{\tau' \in (T_i, T_i + \delta t]} \cdot \mathbb{I}_{\tau > \tau'} (1 - R') \right] \quad (26)
\end{aligned}$$

Assuming independence of the portfolio's value to the probability of either defaults

$$\begin{aligned}
AC_u &= \sum_{i=0}^{N-1} AC_u(T_i) \\
&= \sum_{i=0}^{N-1} B(0, T_i) V^+(T_i) E \left[\mathbb{I}_{\tau' \in (T_i, T_i + \delta t]} \cdot \mathbb{I}_{\tau > \tau'} (1 - R') \right] \quad (27)
\end{aligned}$$

Assuming a deterministic recovery of default

$$\begin{aligned}
AC_u &= \sum_{i=0}^{N-1} AC_u(T_i) \\
&= \sum_{i=0}^{N-1} B(0, T_i) V^+(T_i) (1 - R') E \left[\mathbb{I}_{\tau' \in (T_i, T_i + \delta t]} \cdot \mathbb{I}_{\tau > \tau'} \right] \quad (28)
\end{aligned}$$

The above equation is a summation of N integrals each from T_i to T_{i+1} as δt is still assumed to be infinitesimal. Assuming the independence between the two defaults a modified and a commonly used expression can be obtained by considering discrete times of defaults that fall on T_i s

$$AC_u = (1 - R') \sum_{i=0}^{N-1} B(0, T_i) E \left[V^+(T_i) \right] [P'(T_{i+1}) - P'(T_i)] Q(T_{i+1}) \quad (29)$$

with P and P' to be the cumulative probability of default for the unprimed and primed counterparty, respectively. Note the consequence of survival probability until the end of the period rather than until the default of the primed counterparty, due to discrete default times. The cumulative survival probability is obtained from

$$Q(t) = 1 - P(t) \quad (30)$$

and

$$Q'(t) = 1 - P'(t). \quad (31)$$

Having established (22), since one party's asset charge is the liability benefit of another, the calculation for the liability benefit is only a matter of exercise.

5.2 Derivation of Liability Benefit

In the above section, starting from (12), the asset charge term was derived. From (13) the same method can be applied in order to obtain the liability benefit

$$\begin{aligned} LB_u &= \sum_{i=0}^{N-1} LB_u(T_i) \\ &= \sum_{i=0}^{N-1} B(0, T_i) E [V^-(T_i) \mathbb{I}_{\tau \in (T_i, T_i + \delta t]} \cdot \mathbb{I}_{\tau' > \tau} (1 - R)] \end{aligned} \quad (32)$$

Given the same assumptions, as in the case of the asset charge, a simpler description for Liability Charge can be obtained

$$LB_u = (1 - R) \sum_{i=0}^{N-1} B(0, T_i) E [V^-(T_i)] [P(T_{i+1}) - P(T_i)] Q'(T_{i+1}) \quad (33)$$

Equation (32) represents the liability benefit of the unprimed counterparty to the primed counterparty. Note that switching the prime and unprime sign and changing the sign of (33) one obtains (22). In simple terms, one counterparty's asset charge, is another counterparty's liability benefit.

5.3 A Toy Example: Liability Benefit in an Un-collateralized Account

To get a better sense of what has been achieved so far, consider the case of a LIBOR entity, with 5% hazard rate (λ) selling an nontransferable \$1,000 bullet bond, maturing in one year, to a sub-LIBOR entity with 10% hazard rate (λ'). Without the loss of generality, one can assume the interest rate and the rate of recovery of zero for both counterparties. Though such an example is an over simplification, as a toy example, it provides the basic building block and the ground work for more complicated and realistic situations. Within the context of the example, one can now ask the following question: Given a

mark to market of, say \$1,000, what is the liability benefit that the LIBOR entity should be stating?

For further calculations, $N = 1$, $B(0, 0) = \$1$, $R = R' = 0$, $V^-(0) = -\$1,000$ and $\delta t = 1$. Using the hazard rates provided above, different calculations for the 1-year horizon is given. From (33)

$$\begin{aligned}
LB_u &= (1 - R) \sum_{i=0}^{N-1} B(0, T_i) E [V^-(T_i)] [P(T_{i+1}) - P(T_i)] Q'(T_{i+1}) \\
&= -\$1,000 \times [(1 - e^{-\lambda \times 1}) - 0] \times e^{-\lambda' \times 1} \\
&\cong -\$44.13
\end{aligned} \tag{34}$$

On the hand, using (32),

$$\begin{aligned}
LB_u &= E [-\$1,000 \times \mathbb{I}_{\tau \in (0,1]} \cdot \mathbb{I}_{\tau' > \tau}] \\
&= -\$1,000 \times \int_{\tau'} \int_{\tau} d\mathbb{P}(\tau, \tau') \mathbb{I}_{\tau \in (0,1]} \cdot \mathbb{I}_{\tau' > \tau}
\end{aligned} \tag{35}$$

Assuming independence between τ and τ'

$$\begin{aligned}
LB_u &= -\$1,000 \times \int_{\tau'} \int_{\tau} dP(\tau) dP'(\tau') \mathbb{I}_{\tau \in (0,1]} \cdot \mathbb{I}_{\tau' > \tau} \\
&= -\$1,000 \times \int_0^1 \int_{\tau}^{\infty} dP(\tau) dP'(\tau') \\
&= -\$1,000 \times \int_0^1 \int_{\tau}^{\infty} \lambda \exp(-\lambda t) dt \lambda' \exp(-\lambda' t') dt' \\
&= -\$1,000 \times \lambda \int_0^1 e^{-(\lambda + \lambda')t} dt \\
&= -\$1,000 \times \frac{\lambda}{(\lambda + \lambda')} (1 - e^{-(\lambda + \lambda') \times 1}) \\
&\cong -\$46.43
\end{aligned} \tag{36}$$

6 Decomposition of a Collateralized Portfolio

In the case of un-collateralized portfolio, the amount at risk was the market value of the portfolio. The counterparty with the positive value had the positive exposure to the defaulting counterparty. In the case of the collateralized portfolio, the exposure is reduced due to the collateral. Among other issues, a collateral arrangement also defines the margining period which theoretically

sets a fixed schedule on which applicable collateral can be called. The risk, conditional on both counterparties surviving until the last margin call date, would be the excess of the portfolio value over the collateral held due the most *recent* collateral call. The simplest approach to collateralized exposure is to look at the collateral process separate from the portfolio value. In this method the most effort is focused on the simulation of the collateral. To start with, the collateralized portfolio can be offset by the collateral amount as

$$\begin{aligned} V_s(t, \tau, \tau') &\equiv \{V(t) - C(t_c)G(t_c, t)\} [\mathbb{I}_{\tau' > t} \cdot \mathbb{I}_{\tau > t}] \\ &= V_s^+(t, \tau, \tau') \mathbf{1}^+ - V_s^-(t, \tau, \tau') \mathbf{1}^- \end{aligned} \quad (37)$$

where t_c belongs to the last collateral call prior to t . The amount $C(t_c)$ is the collateral received (+) or posted (-) at t_c . The factor $G(t_c, t)$ is the growth factor active from the last collateral call until t . It is usually pre defined in CSA as to how it is calculated. In most cases it is based on overnight rate. Once the portfolio value is adjusted by the collateral, the treatment of the decomposition of this portfolio follows naturally in the same manner as the un-collateralized portfolio. Practically, the main portion of the counterparty risk modeling of a collateralized portfolio is the modeling of the collateral process which strongly depends on how much sophistication is required. To incorporate the threshold level with no minimum transfer amount, the collateral from the unprimed counterparty's view can be modeled as

$$C(t_c) = (V(t_c) - D_r')^+ - (V(t_c) - D_r)^+ \quad (38)$$

where the thresholds D_r' and D_r correspond to the primed and the unprimed counterparties, respectively, and are always positive. To model the unilateral agreement simply assign infinity to D_r' (primed counterparty calls only) or D_r (unprimed counterparty calls only). The sub-index r represents the dependence of the threshold to the rating of the counterparties; a common schedule in CSA agreement. Note that the above equation does not depend on τ nor τ' . This is to reflect the collection of collateral conditional on both counterparties *surviving*. Readers should note that t_c is the last call date *prior* to t . In most cases $V(t_c)$ is simply interpolated. The definition of the collateral given in (38) is different from what is given in [6] since here both terms are given to reflect the fair value and not just the exposure. One can modify the above equation to include the minimum transfer amount

and obtain

$$\begin{aligned}
C(t_c) &= \frac{(V(t_c) - (D'_r + m))^+}{(V(t_c) - (D'_r + m))} (V(t_c) - (D'_r)) \\
&\quad - \frac{(V(t_c) - (D_r + m))^+}{(V(t_c) - (D_r + m))} (V(t_c) - (D_r))
\end{aligned} \tag{39}$$

Obviously, there is no end as to how far the sophistication of the collateral modeling can reach. Margin periods vary in range starting from one day (daily margin) to six months (semi-annual) with daily margin being the most common. While the grace period depends on the regional governing law and it is about two to three business days.

The remaining challenge is to properly calculate the collateral during the life of the portfolio as measured by \mathcal{F}_{T_i} conditional on both survivals. Note that the calculation of the exposure on every margin day is challenging if not impossible. A common practice is to select a set of horizon times that is dense at the beginning and coarse at far horizons. Value of the portfolio at both or either time horizons can then be interpolated using an appropriate interpolation scheme.

To give a flavor of the process, one can follow the following steps: break the entire time life of the portfolio in to discrete time points T_i . For every time horizon T_i calculate $V(T_i)$. Interpolate $V(t_c)$ from $V(T_i)$ and $V(T_{i-1})$. Calculate $C(t_c)$ in (39) and replace it, along with $V(T_i)$, in (37) and use (12) through (15) and the subsequent equations to calculate the CVA. In cases where T_i is smaller than the margin period, $t_c = 0$. That is, $C(0)$ is the collateral currently held/posted.

6.1 A Toy Example: Calculation of Exposure in A Collateralized Account

To get an intuition around collateralized exposure a scenario analysis is given here where the interest rate on cash deposits, threshold level and minimum transfer amount are 0 and there has not been any need for collateral transfer. Assume that at time t_0 the value of the portfolio is +1 to the unprimed counterparty. The unprimed counterparty's exposure at time t_0 is, therefore, +1. Further more, assume that immediately after the evaluation at t_0 , a collateral transfer of +1 is made from the primed counterparty. During the next margin period, within the scope of this scenario, the market moves in such a way that the portfolio is valued at -1 to the unprimed counterparty

and +1 to the primed counterparty (due to some cashflow movement). So far, the total collateral collected by the primed counterparty is -1 (since it has already paid the collateral during the past period) with the value of the portfolio now at +1, the primed counterparty's exposure is now +2.

One important point from the above illustration is the strong dependency of the exposure on the path of the scenario. The next concept to note is that the final spot exposure, +2, was more than the spot value of the portfolio, +1, to the primed counterparty. This differential at risk is commonly called collateral call-back risk. It points to the collateral that the primed counterparty had with the unprimed counterparty that could be lost, and not be "called back", due to the primed counterparty's default.

6.2 Remaining Risks in Current Collateralized Arrangements

The example in the previous section underlines two important elements of the collateralized relationship which are often overlooked. The first element is that in such a relationship two types of payments take place. One type of payment is due to trade payoff which can happen any time according to the trade specification. Another, is the payment due to collateral payable under the collateral agreement on collateral call dates only. The second important element is that the collateral payment lags behind on covering the exposure. These two elements can be a very risky combination. Imagine the case where a large payment is due to the client based on some trade payoff, such as a payment of an FX swap, today. The dealer is not entitled to offset the payment with the *up-coming* required collateral. The dealer has to transfer the payment first and wait until the next immediate call date (at the closest, the next day) in order to get the collateral amount. For as long as the dealer is waiting for the call date and arrival of the collateral, it has undoubtedly an unsecured risk against this client. The probability of the client defaulting on the next day after the trade payment seems small, initially. However, incentive of a defaulting client to survive until the payment is received is directly proportional to the amount of the payment. That is, the client would continue to pay the collateral payments (albeit difficult) to avoid default, receive the long-awaited payment, *then* default on collateral payment and use the proceedings from the payment for restructuring or to increase the recovery rate of its bond holders. The larger the payment, the less the probability of client's default would be continuous over the payment date. Many dealers reason, based on their existing ongoing monitoring of their clients' credit

health, that they would withhold payments to those with deteriorating credit. Unfortunately, this argument could only be true for some clients (such those of publicly trading entities where there are usually more information available in the market place) and does not always apply to highly leveraged and secretive clients like hedge funds.

7 Counterparty Risk and Counterparty Trading

All the calculations regarding the risk management of an existing portfolio, mentioned in this article, are essential. However, the most important risk management strategy is one that signals *before* entering into an unwanted trade. Therefore, the first line of defence is the in-depth knowledge of the counterparties' credit status. The next relevant risk mitigant, equally important, is a strongly binding legal documentation governing the trading relationship. All major dealers, and trading firms, have a legal and a supervisory department with the objective of counterparty risk analysis where credit health and the firm's ongoing exposures to each of its counterparties is analyzed. Each trade, through different mechanism, is then passed through an approval process by this department. This process makes the dynamics of the counterparty risk management a prior-to-trade activity and minimizes the possibility of entering into an excessively risky trade with an unwarranted counterparty. In reality, however, a trade needs to be done and the risk needs to be taken for the simple reason that the dealers are in the business of taking appropriate risks for the appropriate rewards. In such cases the extinguishing trades, with the strongly legally binding documentation, should be the most important risk management strategy. An "extinguisher" trade is one that extinguishes (cancels) once an event, most commonly a credit event, happens. An "extinguisher" trade with a credit trigger, such as credit rating drop, well defined inside each trade confirm, reduces the burden of the credit management of a trading portfolio to a difficult counterparty. This is particularly very suitable for counterparties where the probability of default is very difficult to measure. However, based on the same token, pricing of these trades would remain a challenge and is usually reflected in the spread of the trade.

Where there is risk of loss, there is also the possibility of reward. Roughly speaking, both components of CVA can be monetized by entering into other trades where the option-to-default is cashed-in through upfront premium or higher spread levels [7].

Most trading firms with large counterparty exposures have now a unit with a mandate of monetizing the profit opportunities arising from counterparty credit risk. The desk, in general, acts as a separate dealer providing the CVA risk management for different desks for a fee. In another simple form, it can purchase the asset charge from different desks, with initial profit and manage the exposure thereafter. The desks, on the other hand, are also willing to trade away the credit risk in order to lower their counterparty exposure for further trading opportunities.

To get an intuition around monetizing CVA components, consider the example provided in section 4. In order for the sub-LIBOR counterparty to monetize its liability benefit, it can enter into a back-to-back trade with a *new* LIBOR counterparty, highly correlated with the initial LIBOR counterparty, where it receives the same positive cash flow. However, this trade can have a provision to provide an option to this new counterparty to cancel the trade in case the sub-LIBOR counterparty defaults. This option will worth the same amount as the liability benefit that was received from the original trade.

In reality, the trade provided in section 4 does not easily happen as the original LIBOR quality dealer, who was penalized the asset charge, had no incentive to enter into the trade. However, in more involved transactions liability benefit opportunities are present and need to be monetized.

8 Summary

A basic and introductory review of counterparty valuation adjustment was given. The components to the CVA were derived by decomposing the portfolio's value into a set of binary states.

The application of the formulae depends strongly on the assumptions, sophistication of the simulation and the goodness of the input values. Practically speaking, CVA calculations end up to be crude, mostly due to the lack of proper input values. Except a number of entities, there are not many tradable securities, such as credit default swaps (CDS), reflecting the market's view on credit health of the counterparties. Even when there are such instruments, there is usually a lack of market implied volatilities for them. Another important issue is the interdependence of the processes involved which is usually omitted all together due to insufficient market implied correlation factors.

It is important to note that the discussion in this article was based on a single portfolio. However, in reality, there are as many portfolios, on a dealer's book, as there are active counterparties. Once the error is introduced for a

single portfolio due to lack of proper input value, the accumulation of the errors and the interdependence of the portfolios is still to be tackled. These mentioned issues are real and do cost dealers and trading firms. However, the purpose of itemizing them here is to encourage and motivate the readers for future work and to represent opportunities for further developments.

9 Acknowledgements

Authors are grateful to Alfredo Bequillard for constructive comments and Fong Liu for his continuous support.

References

- [1] International Swaps and Derivatives Association, (2008). *ISDA Market Survey results (Excel)*.
<http://www.isda.org/statistics/historical.html>.
- [2] Michael Pykhtin, Counterparty Credit Risk Modelling: Risk Management Pricing and Regulation, *Risk Books* , 2005.
- [3] Paul C. Harding, Mastering the ISDA Master Agreements, *FT. Prentice Hall*, 2002.
- [4] Damiano Brigo and Fabio Mercurio, Interest Rate Models - Theory and Practice with Smile, Inflation and Credit *Springer*, 2006, **748-750**.
- [5] Eduardo Canabarro and Darrell duffie, Chapter 9: Measuring and Marking Counterparty Risk, ALM of financial Institutions, *Institutional Investor Books*, 2004, **Exhibit-9.2**.
- [6] Michael S. Gibson. Measuring Counterparty Credit Exposure to a Margined Counterparty. *U.S Federal Reserve working paper*, 2005, **50**.
- [7] Yi Tang and Bin Li. Quantitative Analysis, Derivative Modeling and Trading Strategies. *World Scientific*, 2007, **chapter-1**.